

PARITY CONDITION FOR IRREDUCIBILITY OF HEEGAARD SPLITTINGS

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ABSTRACT. Casson and Gordon gave the rectangle condition for strong irreducibility of Heegaard splittings [1]. We give a parity condition for irreducibility of Heegaard splittings of irreducible manifolds. As an application, we give examples of non-stabilized Heegaard splittings by doing a single Dehn twist.

1. INTRODUCTION

Given a non-minimal genus Heegaard splitting of a 3-manifold, it is not an easy problem to show that it is irreducible or cannot be destabilized. In [3], Kobayashi showed that every genus $g \geq 3$ Heegaard splitting of 2-bridge knot exterior is reducible. The motivation of this paper was the question that whether there exists an irreducible genus three Heegaard splitting of a tunnel number one knot exterior which is not 2-bridge.

Casson and Gordon used the rectangle condition on Heegaard diagrams to show strong irreducibility of certain manifolds obtained by surgery in their unpublished paper [1]. See also ([5], Appendix). One can also refer to the paper by Kobayashi [2] and Saito ([6], section 7) for a good application of the rectangle condition.

In section 3, we review that Casson-Gordon's rectangle condition implies strong irreducibility of Heegaard splittings. Also we consider weak version of rectangle condition for manifolds with non-empty boundary.

In section 4, we give a parity condition on Heegaard diagrams to guarantee that the given Heegaard splitting is non-stabilized. Hence, if the manifold under consideration is irreducible, the Heegaard splitting is irreducible.

Theorem 1.1. *Suppose M is an irreducible 3-manifold. Let $H_1 \cup_S H_2$ be a Heegaard splitting of M . Let $\{D_1, D_2, \dots, D_{3g-3}\}$ and $\{E_1, E_2, \dots, E_{3g-3}\}$ be collections of essential disks of H_1 and H_2 respectively giving pants decomposition of S .*

If $|D_i \cap E_j| \equiv 0 \pmod{2}$ for all the pairs (i, j) , then $H_1 \cup_S H_2$ is irreducible.

As an application, we give examples of non-stabilized Heegaard splittings by doing a single Dehn twist, in section 5.

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2. PANTS DECOMPOSITION AND ESSENTIAL DISK IN A HANDLEBODY

Let H be a genus $g \geq 2$ handlebody. Suppose a collection of $3g - 3$ essential disks $\{D_1, D_2, \dots, D_{3g-3}\}$ cuts H into a collection of $2g - 2$ solid pants-shaped 3-balls $\{B_1, B_2, \dots, B_{2g-2}\}$. Let P_i be the pants $B_i \cap \partial H$ ($i = 1, 2, \dots, 2g - 2$). Then $P_1 \cup P_2 \cup \dots \cup P_{2g-2}$ is a *pants decomposition* of ∂H .

Let D be an essential disk in H . Assume that D intersects $\bigcup_{i=1}^{3g-3} D_i$ minimally.

Definition 2.1. A wave $\alpha(D)$ for an essential disk D and the collection of essential disks $\{D_1, D_2, \dots, D_{3g-3}\}$ of a handlebody H is a subarc of ∂D cut by the intersection $D \cap (\bigcup_{i=1}^{3g-3} D_i)$ satisfying the following conditions.

- There exist an outermost arc β and outermost disk Δ of D with $\partial\Delta = \alpha(D) \cup \beta$.
- $(\alpha(D), \partial\alpha(D))$ is not homotopic, in ∂H , into ∂D_i containing $\partial\alpha(D)$.

Lemma 2.2. Suppose D is not isotopic to any D_i ($i = 1, 2, \dots, 3g - 3$). Then ∂D contains a wave.

Proof. Suppose $D \cap (\bigcup_{i=1}^{3g-3} D_i) = \emptyset$. Then ∂D lives in one of the pants P_i for some i . Hence we can see that D is isotopic to some D_i , which contradicts the hypothesis of lemma. Therefore $D \cap (\bigcup_{i=1}^{3g-3} D_i) \neq \emptyset$.

Consider the intersection $D \cap (\bigcup_{i=1}^{3g-3} D_i)$. We can eliminate simple closed curves of intersection in D by standard innermost disk argument since handlebodies are irreducible. So we may assume that the intersection is a collection of arcs. Then there always exists an outermost arc and outermost disk, hence a wave. \square

The collection of arcs of intersection $D \cap (\bigcup_{i=1}^{3g-3} D_i)$ divides D into subdisks. A subdisk would be a $2n$ -gon such as bigon, 4-gon, and so on. Note that bigons are in one-to-one correspondence with outermost disks, hence with waves. The following observation is important for the parity condition that will be discussed in section 4.

Lemma 2.3. For all i ($i = 1, 2, \dots, 3g - 3$), $|\partial D \cap \partial D_i| \equiv 0 \pmod{2}$.

Proof. Since any arc of intersection of $D \cap D_i$ has two endpoints, $|\partial D \cap \partial D_i|$ would be an even number. \square

3. RECTANGLE CONDITION

Let S be a closed genus $g \geq 2$ surface and P_1 and P_2 be pants in S with $\partial P_i = a_i \cup b_i \cup c_i$ ($i = 1, 2$). Assume that ∂P_1 and ∂P_2 intersect transversely.

Definition 3.1. *We say that P_1 and P_2 are tight if*

- (1) *There is no bigon Δ in S with $\partial\Delta = \alpha \cup \beta$, where α is a subarc of ∂P_1 and β is a subarc of ∂P_2*
- (2) *For the pair (a_1, b_1) and (a_2, b_2) , there is a rectangle R embedded in P_1 and P_2 such that the interior of R is disjoint from $\partial P_1 \cup \partial P_2$ and the edges of R are subarcs of a_1, b_1, a_2, b_2 . This statement holds for the following all nine combinations of pairs.*

$$\begin{array}{lll} (a_1, b_1), (a_2, b_2) & (a_1, b_1), (b_2, c_2) & (a_1, b_1), (c_2, a_2) \\ (b_1, c_1), (a_2, b_2) & (b_1, c_1), (b_2, c_2) & (b_1, c_1), (c_2, a_2) \\ (c_1, a_1), (a_2, b_2) & (c_1, a_1), (b_2, c_2) & (c_1, a_1), (c_2, a_2) \end{array}$$

Let $H_1 \cup_S H_2$ be a genus $g \geq 2$ Heegaard splitting of a 3-manifold M . Let $\{D_1, D_2, \dots, D_{3g-3}\}$ be a collection of essential disks of H_1 giving a pants decomposition $P_1 \cup P_2 \cup \dots \cup P_{2g-2}$ of S and $\{E_1, E_2, \dots, E_{3g-3}\}$ be a collection of essential disks of H_2 giving a pants decomposition $Q_1 \cup Q_2 \cup \dots \cup Q_{2g-2}$ of S . Casson and Gordon introduced the rectangle condition to show strong irreducibility of Heegaard splittings [1]. See also [2].

Definition 3.2. *We say that $P_1 \cup P_2 \cup \dots \cup P_{2g-2}$ and $Q_1 \cup Q_2 \cup \dots \cup Q_{2g-2}$ of $H_1 \cup_S H_2$ satisfies the rectangle condition if for each $i = 1, 2, \dots, 2g - 2$ and $j = 1, 2, \dots, 2g - 2$, P_i and Q_j are tight.*

At a first glance, it looks not so obvious that rectangle condition implies strong irreducibility. Here we give a proof.

Proposition 3.3. *Suppose $P_1 \cup P_2 \cup \dots \cup P_{2g-2}$ and $Q_1 \cup Q_2 \cup \dots \cup Q_{2g-2}$ of $H_1 \cup_S H_2$ satisfies the rectangle condition. Then it is strongly irreducible.*

Proof. Suppose $H_1 \cup_S H_2$ is not strongly irreducible. Then there exist essential disks $D \subset H_1$ and $E \subset H_2$ with $D \cap E = \emptyset$. Suppose there is a bigon Δ in S with $\partial\Delta = \alpha \cup \beta$, where α is a subarc of ∂D and β is a subarc of ∂P_i for some i . If any subarc of ∂E is in Δ , we remove it by isotopy into $S - \Delta$ before we remove the bigon Δ by isotopy of neighborhood of α in D . So we can remove such bigons maintaining the property that $D \cap E = \emptyset$. Also note that the number of components of intersection $|\partial E \cap (\bigcup_{j=1}^{3g-3} \partial E_j)|$ does not increase after the isotopy since there is no bigon Δ' with $\partial\Delta' = \gamma \cup \delta$, where γ is a subarc of ∂P_i for some i and δ is a subarc of ∂Q_j for some j by the definition of tightness of P_i and Q_j . We can also remove a bigon made by a subarc of ∂E and a subarc of ∂Q_j for some j similarly. So we may assume

that D intersects $(\bigcup_{i=1}^{3g-3} D_i)$ minimally and E intersects $(\bigcup_{j=1}^{3g-3} E_j)$ minimally with $D \cap E = \emptyset$.

Suppose D is isotopic to D_i for some i . Let $\partial Q_j = a_j \cup b_j \cup c_j$ ($j = 1, 2, \dots, 2g-2$). Then $\partial D \cap Q_j$ contains all three types of essential arcs $\alpha_{j,ab}, \alpha_{j,bc}, \alpha_{j,ca}$ by the rectangle condition, where $\alpha_{j,ab}$ is an arc in Q_j connecting a_j and b_j , $\alpha_{j,bc}$ is an arc connecting b_j and c_j and $\alpha_{j,ca}$ is an arc connecting c_j and a_j . Then E is not isotopic to any E_j since $D \cap E = \emptyset$. Then ∂E contains a wave by Lemma 2.2 and this is a contradiction since a wave intersects at least one of $\alpha_{j,ab}, \alpha_{j,bc}, \alpha_{j,ca}$ for some j and $D \cap E = \emptyset$.

If D is not isotopic to any D_i , ∂D contains a wave by Lemma 2.2. Then also in this case, $\partial D \cap Q_j$ contains all three types of essential arcs $\alpha_{j,ab}, \alpha_{j,bc}, \alpha_{j,ca}$ of Q_j by the rectangle condition. This gives a contradiction by the same argument as in the above. \square

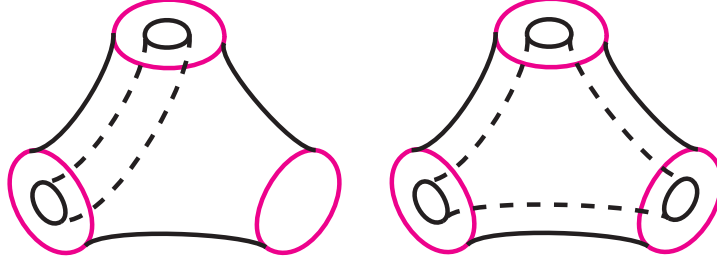


FIGURE 1. Pants decomposition for compressionbody

Now we consider manifolds with non-empty boundary. For a given compressionbody, there can be many ways of pants decomposition. But we can give a specific pants decomposition as in the Figure 1. where if a piece has inner hole, it is one of the two types in the Figure 1.

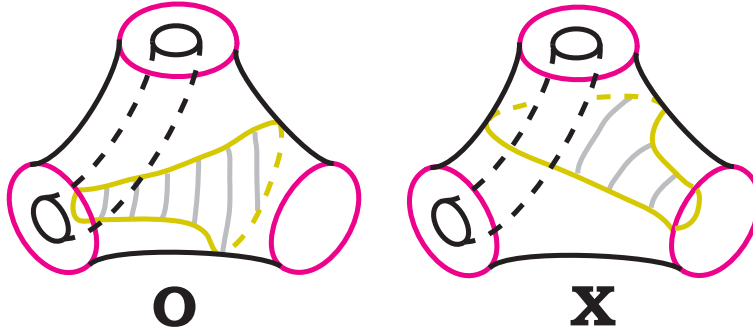


FIGURE 2. Wave is more restrictive in compressionbody.

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For an essential disk in a compressionbody, the existence of wave is more restrictive because of the “minus” boundary of compressionbody (Figure 2.). So we can say the following proposition.

Proposition 3.4. *For a Heegaard splitting $H_1 \cup_S H_2$ and pants decomposition $P_1 \cup P_2 \cup \cdots \cup P_{2g-2}$ and $Q_1 \cup Q_2 \cup \cdots \cup Q_{2g-2}$ of a 3-manifold with non-empty boundary, strictly less number of rectangles compared to Casson-Gordon’s rectangle condition implies strong irreducibility.*

4. PARITY CONDITION

Let $H_1 \cup_S H_2$ be a genus $g \geq 2$ Heegaard splitting of a 3-manifold M . Let $\{D_1, D_2, \dots, D_{3g-3}\}$ and $\{E_1, E_2, \dots, E_{3g-3}\}$ be collections of essential disks of H_1 and H_2 respectively giving pants decomposition of S .

Definition 4.1. *We say that $H_1 \cup_S H_2$ satisfies the even parity condition if $|D_i \cap E_j| \equiv 0 \pmod{2}$ for all the pairs (i, j) .*

First we give an example of an irreducible manifold which has a Heegaard diagram satisfying the even parity condition. The example is a weakly reducible genus three Heegaard splitting of $(\text{torus}) \times S^1$ which gave inspiration for the parity condition. Since $\pi_1((\text{torus}) \times S^1)$ has rank 3, genus three Heegaard splitting is of minimal genus, hence irreducible.

Consider $(\text{torus}) \times \{\text{pt}\}$ in $(\text{torus}) \times S^1$. Remove small open disk $\text{int}(D)$ from $(\text{torus}) \times \{\text{pt}\}$ and take its product neighborhood $((\text{torus}) \times \{\text{pt}\} - \text{int}(D)) \times [0, 1]$. It is a genus two handlebody. Attach a 1-handle to $((\text{torus}) \times \{\text{pt}\} - \text{int}(D)) \times [0, 1]$ along two disk $D_1 \subset ((\text{torus}) \times \{\text{pt}\} - \text{int}(D)) \times \{0\}$ and $D_2 \subset ((\text{torus}) \times \{\text{pt}\} - \text{int}(D)) \times \{1\}$. The result is a genus three handlebody $H_1 = ((\text{torus}) \times \{\text{pt}\} - \text{int}(D)) \times [0, 1] \cup_{D_1 \cup D_2} 1\text{-handle}$. The exterior $H_2 = \text{cl}((\text{torus}) \times S^1 - H_1)$ is also a genus three handlebody and this gives a Heegaard splitting $(\text{torus}) \times S^1 = H_1 \cup H_2$. This kind of Heegaard splitting is well understood, for example in [7].

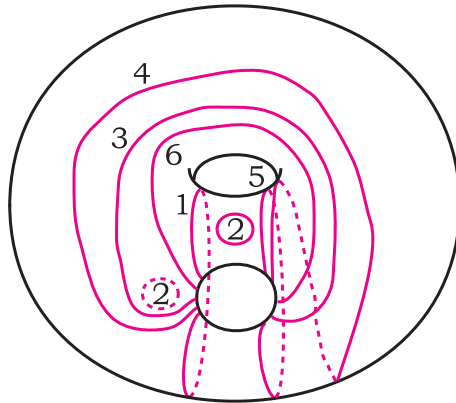


FIGURE 3. essential disks in H_1 giving a pants decomposition

Take a collection of essential disks with labels $1, 2, 3, 4, 5, 6$ in H_1 giving a pants decomposition as in the Figure 3. Figure 3. shows essential arcs $1, 3, 4, 5, 6$ in a once punctured torus, but regard the surface and curves as being taken products with $[0, 1]$. The two curves with label 2 indicates the positions to which a 1-handle is attached.

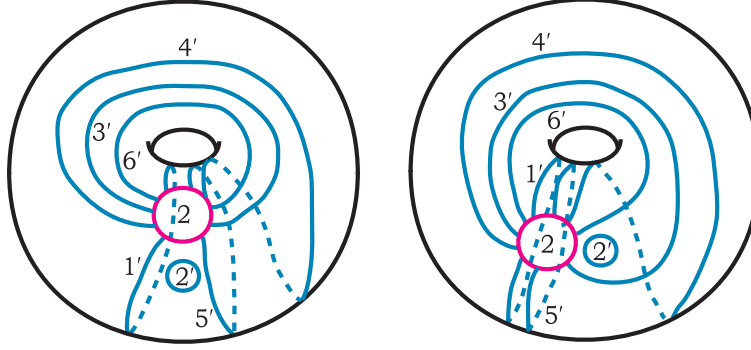


FIGURE 4. Intersections of essential disks $1', 2', 3', 4', 5', 6'$ of H_2 with $(\text{torus}) \times \{0\}$ and $(\text{torus}) \times \{1\}$

In H_2 , a collection of essential disks with labels $1', 2', 3', 4', 5', 6'$ giving a pants decomposition is taken as in the Figure 4. The left one in Figure 4. is $H_2 \cap ((\text{torus}) \times \{0\})$ and the right one is $H_2 \cap ((\text{torus}) \times \{1\})$. Each of arcs $1', 3', 4', 5', 6'$ in the left one is connected to corresponding one in the right passing through the disk with label 2. A 1-handle is attached along the disks with label 2'.

Now we have information about the boundaries of all 12 essential disks of this specific decomposition and we can check all pairs of intersections. For each of disks $1', 2', 3', 4', 5', 6'$, the sequence of intersections with $1, 2, 3, 4, 5, 6$ is as follows ignoring the orientations and starting points.

$$1' = 1436123462$$

$$2' = 1546351436$$

$$3' = 1452341532$$

$$4' = 1635261532$$

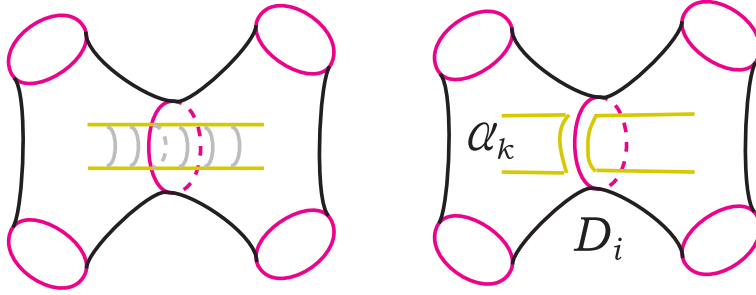
$$5' = 5364523462$$

$$6' = 1452615462$$

So we can see that it satisfies the even parity condition, hence irreducible.

Proof. (of Theorem 1.1) It is well known that reducible Heegaard splitting of an irreducible manifold is stabilized. So it suffices to show that $H_1 \cup_S H_2$ is non-stabilized.

Suppose that $H_1 \cup_S H_2$ is stabilized. Then there exist essential disks D in H_1 and E in H_2 such that $|D \cap E| = 1$. We may assume that the intersection


 FIGURE 5. α_k lives in a pair of pants.

$D \cap (\bigcup_{i=1}^{3g-3} D_i)$ is a collection of arcs. Cut D by $(\bigcup_{i=1}^{3g-3} D_i)$. Then D is divided into subdisks. For any arc, say γ , of intersection $D \cap D_i$, two copies of γ , γ_1 and γ_2 are created on both sides of D_i which are parallel to each other. Connect two endpoints of γ_1 and also connect two endpoints of γ_2 by arcs in S that are parallel and in opposite sides of D_i to each other as in the Figure 5. We do this cut-and-connect operation for all the arcs $D \cap (\bigcup_{i=1}^{3g-3} D_i)$. Let $\{\alpha_k\}$ be a collection of loops thence obtained from ∂D . Note that each α_k lives in a pair of pants. Some α_k would be isotopic to ∂D_{i_k} and some other α_k would possibly be a trivial loop.

Similarly, from ∂E we obtain a collection of loops $\{\beta_k\}$. Some β_k would be isotopic to ∂E_{j_k} and some other β_k would possibly be a trivial loop.

First we consider the parity of $|\partial D \cap \partial E_j|$ for each j ($j = 1, 2, \dots, 3g - 3$) which will be used in the below. Its parity is equivalent to $\sum_k |\alpha_k \cap \partial E_j| \pmod{2}$ since in the above cut-and-connect operation two parallel copies γ_1 and γ_2 were created. It is again equivalent to $\sum_k |\partial D_{i_k} \cap \partial E_j| + \sum |(\text{trivial loop}) \cap \partial E_j| \pmod{2}$. By the hypothesis of even parity condition, it is even. By similar arguments, we have the following equalities in $\pmod{2}$.

$$|\partial D \cap \partial E| \equiv \sum_k |\partial D \cap \beta_k| \equiv \sum_k |\partial D \cap \partial E_{j_k}| + \sum |\partial D \cap (\text{trivial loop})| \equiv 0$$

This is a contradiction since $|D \cap E| = 1$. So we conclude that $H_1 \cup_S H_2$ is irreducible. \square

5. EXAMPLES OBTAINED BY A SINGLE DEHN TWIST

It is known that there exist 3-manifolds which have infinitely many distinct strongly irreducible Heegaard splittings. Many of the known examples are obtained from a given Heegaard splitting by Dehn twisting the surface several times. For example, see [1] and [4]. In this section, we obtain

non-stabilized Heegaard splittings by just a single Dehn twist, if the given splitting satisfies the even parity condition.

Lemma 5.1. *Suppose a genus $g \geq 2$ Heegaard splitting $H_1 \cup_S H_2$ and collections of essential disks $\{D_1, D_2, \dots, D_{3g-3}\}$ and $\{E_1, E_2, \dots, E_{3g-3}\}$ satisfy the even parity condition. Let α be a simple closed curve in S such that $|\alpha \cap E_j| \equiv 0 \pmod{2}$ for all $(j = 1, 2, \dots, 3g - 3)$.*

If we alter $\{D_1, D_2, \dots, D_{3g-3}\}$ to $\{D'_1, D'_2, \dots, D'_{3g-3}\}$ by Dehn twisting ∂H_1 along α and don't change $\{E_1, E_2, \dots, E_{3g-3}\}$, then $\{D'_1, D'_2, \dots, D'_{3g-3}\}$ and $\{E_1, E_2, \dots, E_{3g-3}\}$ satisfy the even parity condition.

Proof. By Dehn twisting ∂H_1 along α , we can calculate that $|D'_i \cap E_j|$ is equal to $|D_i \cap E_j| + |D_i \cap \alpha| \cdot |\alpha \cap E_j|$. Since both $|D_i \cap E_j|$ and $|\alpha \cap E_j|$ are even, $|D'_i \cap E_j|$ is an even number. So we conclude that $\{D'_1, D'_2, \dots, D'_{3g-3}\}$ and $\{E_1, E_2, \dots, E_{3g-3}\}$ satisfy the even parity condition. \square

Take a neighborhood $N(\alpha)$ of α in H_1 . We may assume that $\text{cl}(H_1 - N(\alpha))$ is a handlebody of same genus with H_1 and $\text{cl}(H_1 - N(\alpha)) \cap N(\alpha)$ is an annulus. Remove $N(\alpha)$ from H_1 and attach a solid torus back to $\text{cl}(H_1 - N(\alpha))$ so that a meridian of the attaching solid torus is mapped to $\frac{1}{1}$ -slope of $\partial N(\alpha)$. This $\frac{1}{1}$ -surgery is equivalent to that H_1 is changed by a Dehn twist of ∂H_1 along α . Hence we have the following result by Lemma 5.1.

Theorem 5.2. *Suppose a genus $g \geq 2$ Heegaard splitting $H_1 \cup_S H_2$ and collections of essential disks $\{D_1, D_2, \dots, D_{3g-3}\}$ and $\{E_1, E_2, \dots, E_{3g-3}\}$ satisfy the even parity condition. Let α be a simple closed curve in S such that $|\alpha \cap E_j| \equiv 0 \pmod{2}$ for all $(j = 1, 2, \dots, 3g - 3)$.*

Then the manifold obtained by $\frac{1}{1}$ -surgery on α has a non-stabilized Heegaard splitting which is obtained from S by a Dehn twist along α .

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